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COVERING CYCLES AND k-TERM DEGREE SUMS

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We show that if $\sum_{x\in S} \deg_G x \geq |G|$, for every stable set $S \subseteq V(G)$, $|S|=k$, then the vertex set of G can be covered with $k-1$ cycles, edges or vertices. This settles a conjecture by Enomoto, Kaneko and Tuza.

Let $X \subseteq V(G)$. Denote by $\alpha(X)$ the largest cardinality of a stable set of G contained in X. For $\alpha(X) \geq k$ define

$$\sigma_k(X) = \min \left\{ \sum_{x\in S} \deg_G x : S \subseteq X \text{ is a stable set in } G \text{ and } |S| = k \right\}.$$

We always denote by n the order of G.

In [3] Enomoto, Kaneko and Tuza conjectured that if $\sigma_k(V(G)) \geq n$ or $\alpha(V(G)) < k$ then the vertex set of G can be covered by $k-1$ cycles, edges or vertices. Note that for $k=2$ this statement is just the theorem by Ore [5] on Hamiltonian cycles. The case of $k=3$ was shown by Enomoto, Kaneko, Kouider and Tuza [2].

In this paper we settle the above conjecture in the general case, i.e. for every $k \geq 2$. The basic idea of the proof is to show a more general theorem then the conjecture of Enomoto, Kaneko and Tuza (see Theorem 1). This way we are able to use a stronger inductive hypothesis in our inductive reasoning.

It has to be mentioned here that a weaker statement:

If the minimum degree of a graph is at least n/k then G can be covered by $k-1$ cycles, edges or vertices

was previously considered too. Obviously it is a generalization of a theorem by Dirac [1] (the case $k=2$). For $k=3$ it was shown by Enomoto, Kaneko and Tuza

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[3] and for every $k\geq 2$ by Kouider [4]. The method used in [4] is different than ours and much more complicated.

Denote consecutive vertices of a fixed path P by $p_1,\dots,p_m$. Define $[p_r,p_s[=\{p_r,\dots,p_{s-1}\}$ (in particular $[p_r,p_s[=\emptyset$ when $r\geq s$). The sets $]p_r,p_s]$, $]p_r,p_s[$ and $[p_r,p_s]$ are defined analogously. For $Q\subseteq P-\{p_1\}$ (respectively $Q\subseteq P-\{p_m\}$) let $Q^+=\{p_{i+1}:p_i\in Q\}$ (resp. $Q^-=\{p_{i-1}:p_i\in Q\}$).

Let us start with the following generalization of the Ore theorem.

Lemma 1. *Let G be a graph on n vertices and let $X\subseteq V(G)$. If $\sigma_2(X)\geq n$ or X is a clique then X can be covered with a cycle, an edge or a vertex of G.*

Proof. As, by assumption, X must be contained in one of the components of G, we can suppose that G is connected. Let $P=(p_1,\dots,p_m)$ be a path in G with both ends in X containing the largest possible number of vertices from X. Define $A=(\Gamma_P(p_1))^-$ and $B=\Gamma_P(p_m)$.

We can assume that the set $\{p_1,p_m\}$ is stable and $\Gamma_{G-P}(p_1)\cap\Gamma_{G-P}(p_m)=\emptyset$. Otherwise P is contained in a cycle so, if $X\subseteq P$, we are done or, if $X\not\subseteq P$, P can be extended contradicting to the maximality of P. If $A\cap B=\emptyset$ then

$$\begin{aligned} n\leq \deg_G p_1+\deg_G p_m &= (|A|+|\Gamma_{G-P}(p_1)|)+(|B|+|\Gamma_{G-P}(p_m)|)\\ &= |A\cup B\cup\Gamma_{G-P}(p_1)\cup\Gamma_{G-P}(p_m)|<n \end{aligned}$$

because $p_m\notin A\cup B\cup\Gamma_{G-P}(p_1)\cup\Gamma_{G-P}(p_m)$, a contradiction.

Thus $A\cap B\neq\emptyset$ so P is contained in a cycle. If $X\not\subseteq P$ then, by the connectivity of G, P can be extended, a contradiction with the maximality of P. Hence $X\subseteq P$ is contained in a cycle. ∎

Here is the main result of the paper. Note that for $X=V(G)$ we get the conjecture of Enomoto, Kaneko and Tuza.

Theorem 1. *Let G be a graph on n vertices and let $X\subseteq V(G)$. If $\sigma_k(X)\geq n$ or $\alpha(X)<k$ then X can be covered with $k-1$ cycles, edges or vertices of G.*

Proof. We proceed by induction on k. For $k=1$ the theorem is trivially true and for $k=2$ it is shown in Lemma 1.

We can assume that the minimum degree of a vertex in X is greater than 1. Indeed, suppose $\deg_G z=1$, for some $z\in X$. Denote by y the neighbour of z. One can easily check that either $\sigma_{k-1}(X-\{y,z\})\geq n-1$ or $\alpha(X-\{y,z\})<k-1$. The theorem follows by the inductive hypothesis applied for $X-\{y,z\}\subseteq V(G)-\{z\}$. If $\deg_G z=0$, for some $z\in X$ then obviously either $\sigma_{k-1}(X-\{z\})\geq n-1$ or $\alpha(X-\{z\})<k-1$. We are done again by the induction hypothesis.

Let $P=(p_1,\dots,p_m)$ be a path in G with both ends in X containing the largest possible number of vertices from X. Define $A=(\Gamma_P(p_1))^-$ and $B=(\Gamma_P(p_m))^+$. It suffices to show the theorem in the case when every component of G contains some vertex of X because the components without vertices of X are inessential in our coverings (so they contain no neighbours of X). Consider three cases.

Case 1. G is disconnected.

Let W be some component of G and $X'=X\cap W$. Define p to be the smallest integer such that

$$\sigma_p(X')\geq |W| \quad \text{or} \quad \alpha(X')<p. \tag{1}$$

Since (1) holds for $p=k$, p is well-defined. Obviously $p\geq 2$. Moreover $p<k$ for otherwise $\sigma_{k-1}(X')<|W|$ so for every $x\in X-X'$, $\deg_{G-X'}x=\deg_G x\geq n-\sigma_{k-1}(X')>|G-W|$, a contradiction.

Apply the induction hypothesis for $X'\subseteq V(W)$. By (1) we can cover X' by $p-1$ cycles, edges or vertices. On the other hand, by the minimality of p,

$$\sigma_{p-1}(X')<|W| \quad \text{and} \quad \alpha(X')\geq p-1. \tag{2}$$

Since $\sigma_k(X)\geq n$ or $\alpha(X)<k$, either $\sigma_{k-p+1}(X-X')\geq |G-W|$ or $\alpha(X-X')<k-p+1$. By the induction hypothesis for $X-X'\subseteq G-W$, $X-X'$ can be covered with $k-p$, cycles, edges or vertices. Consequently, X can be covered by $k-1$ cycles, edges or vertices.

Case 2. $A\cap B=\emptyset$ and G is connected.

Define r (resp. s) to be the largest (resp. the smallest) index such that either there is a path from p_1 to p_r (resp. from p_m to p_s) internally disjoint from P or there is an edge with one end in p_r (resp. in p_s) and the other one in A (resp. in B).

Note that for every $x\in(X-P)\cup\,]p_r,p_s[$,

$$\Gamma_G(x)\cap(A\cup\Gamma_{G-P}(p_1))=\emptyset \quad \text{and} \quad \Gamma_G(x)\cap(B\cup\Gamma_{G-P}(p_m))=\emptyset.$$

Otherwise we get a contradiction either with the definitions of r or s or with the maximality of P. Moreover, as G is connected $\Gamma_{G-P}(p_1)\cap\Gamma_{G-P}(p_m)=\emptyset$. Otherwise, for any $z\in\Gamma_{G-P}(p_1)\cap\Gamma_{G-P}(p_m)$, $P\cup z$ forms a cycle C so either P can be extended or all elements of X belong to the cycle C.

Let $G'=G-(A\cup B\cup\Gamma_{G-P}(p_1)\cup\Gamma_{G-P}(p_m))$ and $X'=X-([p_1,p_r]\cup[p_s,p_m])$. Note that $X\cap(\Gamma_{G-P}(p_1))\cup\Gamma_{G-P}(p_m))=\emptyset$ because otherwise P can be extended.

If $\alpha(X')\leq k-3$ then, by the induction hypothesis X' can be covered by $k-3$ cycles, edges or vertices. Otherwise, let $\{x_1,\dots,x_{k-2}\}$ be any stable subset of X'. Since the set $\{x_1,\dots,x_{k-2},p_1,p_m\}\subseteq X$ is stable in G,

$$n\leq\sum_{i=1}^{k-2}\deg_G x_i+\deg_G p_1+\deg_G p_m$$

$$=\sum_{i=1}^{k-2}\deg_{G'}x_i+|A|+|\Gamma_{G-P}(p_1)|+|B|+|\Gamma_{G-P}(p_m)|$$

so

$$\sum_{i=1}^{k-2} \deg_{G'} x_i \geq n - (|A| + |\Gamma_{G-P}(p_1)| + |B| + |\Gamma_{G-P}(p_m)|)$$
$$= n - |A \cup \Gamma_{G-P}(p_1) \cup B \cup \Gamma_{G-P}(p_m)| = |G'|,$$

as $A \cap B = \emptyset$. By the induction hypothesis, X' can again be covered by $k-3$ cycles, edges or vertices. Since $[p_1, p_r] \cup [p_s, p_m] \supseteq X - X'$ can be covered by 2 cycles or edges, the theorem follows.

Case 3. $A \cap B \neq \emptyset$ and G is connected.

Let $x_0 \in A \cap B$. Note that x_0 has no neighbours in $X - P$ for otherwise P can be extended. Define $A_0^+ = (\Gamma_G(x_0))^+ \cap [p_1, x_0[$ and $A_0^- = (\Gamma_G(x_0))^- \cap [x_0, p_m]$. Let t be the smallest integer such that $[p_t, p_m]$ can be covered by one cycle. Denote this cycle by C. Clearly $x_0 \in C$.

Let $G' = G - (A_0^+ \cup A_0^- \cup \Gamma_{G-P}(x_0) \cup \{p_m\})$ and $X' = X - C$.

We claim that, for $x \in X'$,

$$\Gamma_G(x) \cap (A_0^+ \cup A_0^- \cup \Gamma_{G-P}(x_0) \cup \{p_m\}) = \emptyset. \tag{3}$$

Suppose it is not true. Then, for $x \in P - C$, we get a contradiction with the definition of t or the maximality of the path P (see Figure 1) and for $x \notin P$, a contradiction with the maximality of the path P (see Figure 2). We have shown that $\Gamma_G(X') \subseteq V(G')$.

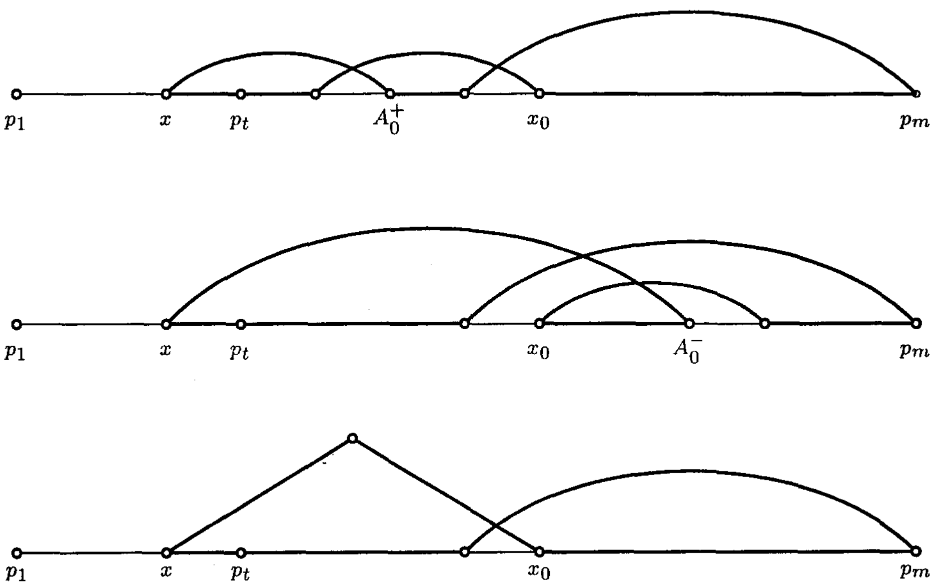

Figure 1

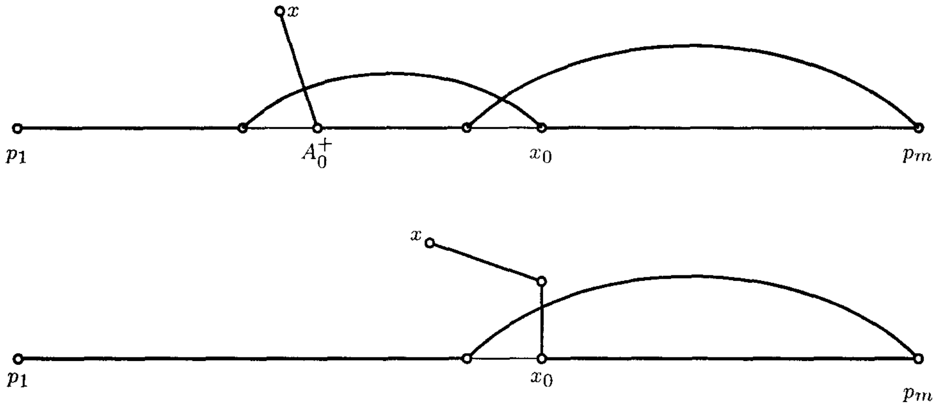

Figure 2

If $\alpha(X') < k-1$ then we are done by induction so suppose that $\alpha(X') \geq k-1$. Let $\{x_1,...,x_{k-1}\}$ be any stable set in X'. The set $\{x_0,x_1,...,x_{k-1}\}$ is stable too by (3) because $x_0 \in A_0^-$. Since $\sigma_k(X) \geq n$,

$$\sum_{i=1}^{k-1} \deg_{G'} x_i = \sum_{i=1}^{k-1} \deg_G x_i \geq n - \deg_G x_0$$
$$= n - |A_0^+ \cup A_0^- \cup \Gamma_{G-P}(x_0) \cup \{p_m\}| = |G'|.$$

By the induction hypothesis X' can be covered by $k-2$ cycles, edges and vertices. Since $X - X' \subseteq C$, the proof is complete. ∎

Analyzing the proof of Theorem 1 one can verify that the following is true.

Theorem 2. *Let G be a 2-edge-connected graph on n vertices and let $X \subseteq V(G)$. If $\sigma_k(X) \geq n$ or $\alpha(X) < k$ then X can be covered with $k-1$ cycles of G.* ∎

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